UK Junior Mathematical Olympiad 2011 Solutions

A1 33 $3^3 + 3 \times 3 = 27 + 9 - 3 = 33$.

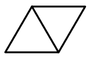
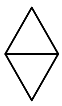
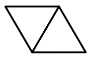
A2 $62\,\text{cm}^2$ Since the area of the 6 cm × 7 cm rectangle is $42\,\text{cm}^2$, the area of the white rectangle is $(42 - 32)\,\text{cm}^2 = 10\text{cm}^2$. Hence the area of the black region is $(8 \times 9 - 10)\,\text{cm}^2 = 62\text{cm}^2$.

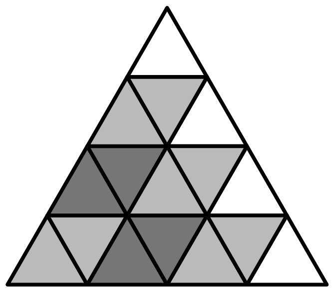
A3 12 In 10 years' time, Paul will be 42 and the sum of the ages of his three sons will be 3×10 years $= 30$ years more than it is now. So the sum of the ages of each of his three sons now is $(42 - 30)$ years $= 12$ years.

A4 10 $\frac{1}{2-3} - \frac{4}{5-6} - \frac{7}{8-9} = \frac{1}{-1} - \frac{4}{-1} - \frac{7}{-1} = (-1) - (-4) - (-7) = -1 + 4 + 7 = 10$.

A5 $n - 1$ A pyramid whose base has n edges also has n edges rising to its apex and hence $2n$ edges in total. It also has $n + 1$ faces, including the base. So the difference between the number of edges of the pyramid and the number of its faces is $2n - (n + 1) = n - 1$.

A6 18 A rhombus formed from a pair of adjacent triangles is in one of three orientations:

 or  or  .



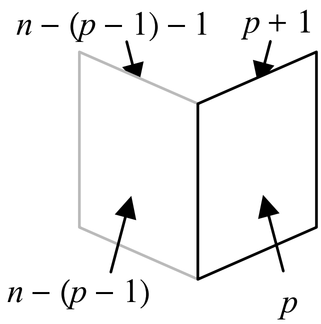
It can be seen that there are 6 rhombi in the first orientation. By symmetry, there are 6 in each of the other two, giving a total of 18.

[*Alternatively*, each rhombus is determined by its short diagonal; the short diagonals are the sides of the interior triangles, so there are $3 \times 6 = 18$.]

A7 7 Let the number of pages in the booklet be n.
Then the page numbers on the outside sheet are 1, 2, $n - 1$ and n and have a total of

$$1 + 2 + (n - 1) + n = 2n + 2.$$

In general, on a sheet whose lowest numbered page is page p, the next page is $p + 1$. Because page p has $(p - 1)$ pages before it, the highest numbered page (on the same side as page p) has $(p - 1)$ pages after it, and so is page $n - (p - 1)$. The remaining page is likewise page $n - (p - 1) - 1$.



The total of the four page numbers of a sheet is therefore

$$p + (p + 1) + [n - (p - 1)] + [n - (p - 1) - 1] = 2n + 2.$$

Hence $2 + 2n = 58$ and so $n = 28$. Since each piece of paper used provides four pages of the booklet, the number of sheets used is seven.

A8 **2** Since we are interested only in whether or not each of the rows can have a total which is a multiple of 3, we can reduce each of the numbers to their remainder on division by three. So

7	5	4
11	10	16
22	19	8

becomes

1	2	1
2	1	1
1	1	2

Since the row totals are 4, 4 and 4, it is now clear that making one swap will not achieve three totals each of which is a multiple of 3. However, if we swap both the 2s in the first two rows with the 1s in the last row, we get

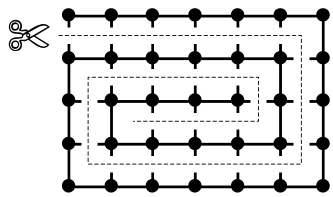
1	1	1
1	1	1
2	2	2

which corresponds to

7	19	4
22	10	16
11	5	8

A9 **24** In order to stay connected, a network like this will have a minimum required number of sections of rope, since if we start with just one knot, each additional knot requires at least one additional section of rope. So the smallest possible number of sections of rope to connect n knots is $n - 1$.

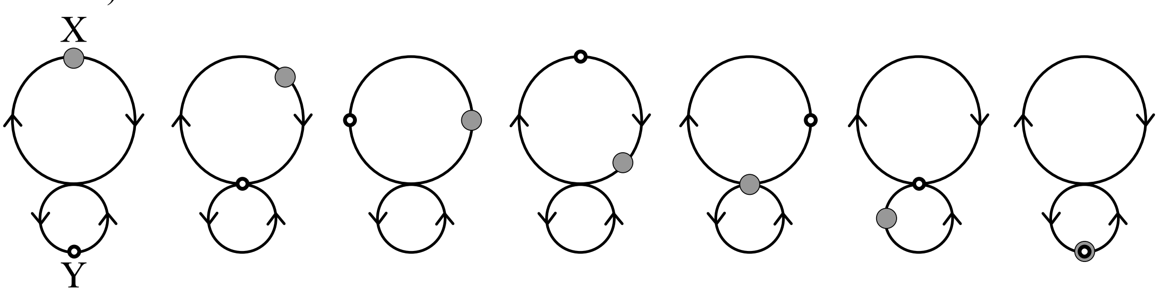
For this net there are 35 knots, and we will need at least 34 sections of rope left to keep them all connected and the diagram on the right shows one way in which we can leave just 34 sections of rope.



It can be seen that the net starts with 5×6 horizontal sections and 7×4 vertical sections, giving a total of $30 + 28 = 58$.

Hence we can make a maximum of $58 - 34 = 24$ cuts.

A10 **3π** The only way for X and Y to collide is by Y catching up X or by both being at the crossing point at the same moment. Since the speed of car X is half that of car Y, car X will travel half way around the circuit in the time car Y takes to complete a full circuit, as the diagrams below illustrate, going from left (the start) to right (the first collision).



The length of the circumference of the larger circle is 4π units, and that of the smaller circle is half as long, 2π units.

Therefore X will have travelled $\frac{1}{2} \times 4\pi + \frac{1}{2} \times 2\pi = 3\pi$ units.

B1 Every digit of a given positive integer is either a 3 or a 4 with each occurring at least once. The integer is divisible by both 3 and 4.

What is the smallest such integer?

Solution

Let the integer be n. It is evident that n must have more than two digits, since none of 3, 4, 34 or 43 are divisible by by both 3 and 4.

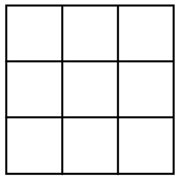
In order for n to be divisible by 3, the sum of its digits must be a multiple of 3.

In order for n to be divisible by 4, its last two digits have to be a multiple of 4. Given that all the digits can be only 3 or 4, it is clear that the last two digits of n can only be 44.

If n were a three-digit number, the hundreds digit would have to be 3 (since the other digits are both 4); however, 344 is not a multiple of 3.

If n were a four-digit number, the smallest value we can consider is 3344, but this is not a multiple of 3. The next smallest is 3444 and this satisfies all the criteria of the problem.

B2 A 3×3 grid contains nine numbers, not necessarily integers, one in each cell. Each number is doubled to obtain the number on its immediate right and trebled to obtain the number immediately below it.



If the sum of the nine numbers is 13, what is the value of the number in the central cell?

Solution

If we let the number in the top left-hand corner cell be a, we can write the other numbers in terms of a, as shown in the diagram:

a	$2a$	$4a$
$3a$	$6a$	$12a$
$9a$	$18a$	$36a$

The sum of these nine numbers is $91a$.

Given that the sum is 13, we have $91a = 13$ and so $a = 13 \div 91 = \frac{1}{7}$.

Thus the number in the central cell is $6a = 6 \times \frac{1}{7} = \frac{6}{7}$.

B3 When Dad gave out the pocket money, Amy received twice as much as her first brother, three times as much as the second, four times as much as the third and five times as much as the last brother. Peter complained that he had received 30p less than Tom.

Use this information to find all the possible amounts of money that Amy could have received.

Solution

As we are considering halves, thirds, quarters and fifths, we shall let Amy receive $60x$ pence.

Then	her first brother receives	$30x$ pence,
	her second brother receives	$20x$ pence,
	her third brother receives	$15x$ pence
and	her last brother receives	$12x$ pence.

We do not know which of the brothers are Peter and Tom, though Peter is younger than Tom. We can now tabulate the six possibilities:

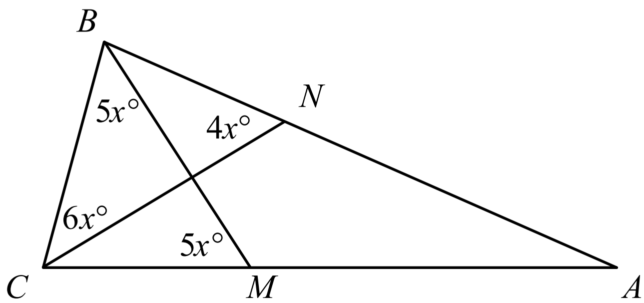
	brothers				comparison	value
case	first	second	third	fourth		of x
A	$30x$	$20x$			$30x - 20x = 30$	$x = 3$
B	$30x$		$15x$		$30x - 15x = 30$	$x = 2$
C	$30x$			$12x$	$30x - 12x = 30$	$x = 1\frac{2}{3}$
D		$20x$	$15x$		$20x - 15x = 30$	$x = 6$
E		$20x$		$12x$	$20x - 12x = 30$	$x = 3\frac{3}{4}$
F			$15x$	$12x$	$15x - 12x = 30$	$x = 10$

We can eliminate two of these cases: in case C, the value of $x = 1\frac{2}{3}$ would mean that the second brother received $20 \times 1\frac{2}{3} = 33\frac{1}{3}$ pence; in case E, the value of $x = 3\frac{3}{4}$ would mean that the third brother received $15 \times 3\frac{3}{4} = 56\frac{1}{4}$ pence.

The remaining four cases mean that Amy could have received £1.20, £1.80, £3.60 or £6.00.

B4 In a triangle ABC, M lies on AC and N lies on AB so that $\angle BNC = 4x°$, $\angle BCN = 6x°$ and $\angle BMC = \angle CBM = 5x°$.

Prove that triangle ABC is isosceles.



Solution

By considering the sum of the angles of triangle NBC, we find that

$$\angle NBC = (180 - 6x - 4x)° = (180 - 10x)°.$$

Similarly, by considering the sum of the angles of triangle MCB, we find that

$$\angle MCB = (180 - 5x - 5x)° = (180 - 10x)°.$$

Hence $\angle ABC = \angle NBC = (180 - 10x)° = \angle MCB = \angle ACB$, and so triangle ABC is isosceles.

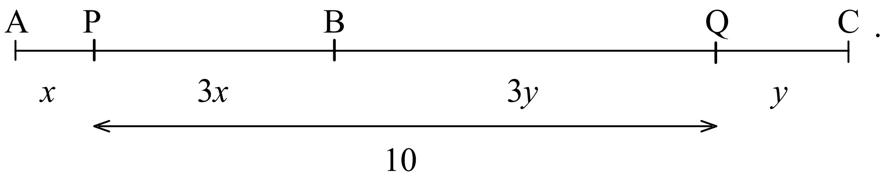
B5 Calum and his friend cycle from A to C, passing through B. During the trip he asks his friend how far they have cycled. His friend replies “one third as far as it is from here to B”. Ten miles later Calum asks him how far they have to cycle to reach C. His friend replies again “one third as far as it is from here to B”.

How far from A will Calum have cycled when he reaches C?

Solution

Let the points at which Calum asked each question be P and Q.
Let the distances AP and QC be x miles and y miles respectively.
Then the distance from P to B is $3x$ miles and the point P must lie between A and B.
Similarly the distance from B to Q is $3y$ miles.

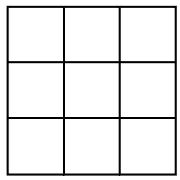
Thus



Since they have cycled 10 miles between P and Q, we know that $3x + 3y = 10$, and so

$$AC = 4x + 4y = \tfrac{4}{3}(3x + 3y) = \tfrac{4}{3} \times 10 = 13\tfrac{1}{3} \text{ miles}.$$

B6 Pat has a number of counters to place into the cells of a 3 × 3 grid like the one shown. She may place any number of counters in each cell or leave some of the cells empty. She then finds the number of counters in each row and each column. Pat is trying to place counters in such a way that these six totals are all different.



What is the smallest total number of counters that Pat can use?

Solution

Let n be the smallest number of counters that Pat can use.

We observe first that n = the sum of the three row totals
= the sum of the three column totals
whence the sum of all six totals = $2n$.

The smallest possible different totals are 0, 1, 2, 3, 4 and 5, so that the least that the sum of the six totals could be is $0 + 1 + 2 + 3 + 4 + 5 = 15$.

Hence $2n \geqslant 15$ and so $n \geqslant 8$.

Consider $n = 8$. It is indeed possible to find a way for Pat to place the counters in such a way that all six totals are all different – two possible configurations are shown, with the totals, on the right:

			0
	1	1	2
1	2	3	6
1	3	4	

			0
	2		2
1	1	4	6
1	3	4	

Therefore the smallest number of counters that Pat can use is 8.